Generalized Instantons on Complex Projective Spaces

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Abstract

We study a class of generalized self-duality relations in gauge theories on the complex projective space with the Fubini-Study metric. Our theories consist of only gauge fields with gauge group $U(n)$. The pseudo-energies which we consider contain higher orders of field strength and are labeled by an integer p smaller than or equal to $[n/2]$. For making the Bogomol'nyi completion we need non-single trace terms in the pseudo-energies, unlike the models defined on spheres, which were studied previously. We construct an explicit solution of codimension $2n$ to generalized self-duality equations as Bogomol'nyi equations, by using a part of the spin connection.

I. INTRODUCTION

The Yang-Mills instantons [1] play important roles in field theory and string theory [2]. The solutions satisfy the (anti)self-duality relation $F = \pm * F$ in four-dimensional Euclidean space, which is a set of non-linear first order partial differential equations. Generalizations of the duality relation in higher dimensional spaces have been discussed in several ways. One way is to find an antisymmetric invariant tensor T of rank 4 on the base space, and to consider the first order relations $F = \pm TF$ where the Hodge duality is not used in general [3]. Another way introduced by Tchrakian [4]–[8] is to consider Hodge dual relations (generalized self-duality relations) between higher order terms of the field strength F. Although these are not of first order anymore, an advantage is that no extra tensor like T is needed. An explicit example of the octonionic instantons satisfying generalized (anti-)self-duality relations $F^2 = \pm * F^2$ was obtained on eight dimensional sphere S^8 [5]. Recently we have studied a monopole-like solution of codimension five in generalized self-duality relations $* F^2 = \pm D_A \phi$ (with ϕ Higgs scalars) as Bogomol'nyi equations in gauge systems with higher derivative couplings [9]. We have also constructed an explicit solution of an "instanton" of codimension six, satisfying generalized self-duality relations $F^2 = \pm \gamma_7 * F$ with SO(6) gauge group on S^6 [10] and have applied it to a compactification $M_4 \times S^6$ or $AdS_4 \times S^6$ [11].

In this paper we study the Tchrakian's type self-duality relations of U(n) gauge theory on general manifolds of dimension $2n$, and construct an explicit solution of codimension $2n$ on the complex projective space $\mathbb{C}P^n$. Since Hermitian matrices are closed in their multiplications, higher powers of the field strength F are elements of Lie algebra , in the case of U(n) gauge group. The previous example of SO(6) is a special case because of the local isomorphism; SO(6)$\simeq_{\text{local}}$ SU(4). We consider pseudo-energies which consist of single and double trace terms of field strength F although one could consider triple or higher trace terms in general. In previously known examples only single trace term was considered. We see that double-trace term is needed in the pseudo-energy in order to make the Bogomol'nyi completion and it is enough to construct a non-trivial topological solution. In this kind of theories, self-duality relation is given as a relation between the Hodge dual of one power of the field strength and another power of it where we may need modification by subtracting trace parts. Such a self-duality relation has been already discussed in [6, 7]. We revise them and find that our solution has non-vanishing pseudo-energy.

Our previous solution on S^6 was applied to compactification $AdS_4 \times S^6$ [11] in the spirit of Cremmer and Scherk [12]. There exists a certain relation between a coupling constant and the radius of S^6, suggesting the stability of S^6. So the solution obtained in this paper may be applied to a compactification such as $AdS_4 \times \mathbb{C}P^3$ which may have some relation to string theory.

This paper is organized as follows. In Sec. II we give a pseudo-energy on general manifolds of even dimensions, which realize self-duality relations among higher order terms of the field strength as Bogomol'nyi equations. In Sec. III we construct an explicit solution in the case that the base manifold is the complex projective space $\mathbb{C}P^n$. Sec. IV is devoted to conclusion and discussion. In Appendix we summarize basic ingredients of the complex projective space $\mathbb{C}P^n$, including Hodge dual relations among higher order terms of the curvature tensor crucially used in this paper.

II.   DUALITY AS BOGOMOL'NYI EQUATION

We study U(n) gauge theory with higher derivative couplings in $2n$ dimensions. First we consider a pseudo-energy introduced by Tchrakian [4]. In this section we work on a general $2n$-dimensional Riemannian manifold M with a metric G, and we will restrict ourselves to the case of $M = \mathbb{C}P^n$ when we consider an explicit solution in the next section. On a generic atlas, we parameterize it by the coordinates x^μ ($\mu = 1, \cdots, 2n$). Gauge fields are represented as a $u(n)$-valued one-form A which is an $n \times n$ matrix with coefficients of differential forms. The field strength two-form F is defined as $F = dA + A \wedge A$. Let T^a ($a = 1, 2, \cdots, g := n^2$) be generators of the Lie algebra $u(n)$. The gauge field A can be expanded by these basis;
$A = A_m^a T^a dx^m$.

We consider U(n) gauge systems whose pseudo-energy is given by a sum of energy functionals

$$\mathcal{E} = \sum_{p=0}^{n} \left(\alpha_p^2 \mathcal{E}_p^{(1)} + \beta_p^2 \mathcal{E}_p^{(2)} \right), \tag{1}$$

with α_p, β_p real positive constants, and energy functionals $\mathcal{E}_p^{(1)}$ and $\mathcal{E}_p^{(2)}$ defined by

$$\mathcal{E}_p^{(1)} = \int_M \mathrm{Tr} \boldsymbol{e}_p(F) \wedge *\boldsymbol{e}_p(F), \qquad \mathcal{E}_p^{(2)} = \int_M \mathrm{ch}_p(F) \wedge *\mathrm{ch}_p(F). \tag{2}$$

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Here $e_p(F)$ and the p-th Chern character $\mathrm{ch}_p(F)$ are defined by

$$e(F) := \exp\left(\frac{\mathbf{i}F}{2\pi}\right) = \sum_{p=0}^{\infty} e_p(F), \qquad e_p(F) := \frac{1}{p!}\left(\frac{\mathbf{i}F}{2\pi}\right)^p$$

$$\mathrm{ch}(F) := \mathrm{Tr}\exp\left(\frac{\mathbf{i}F}{2\pi}\right) = \sum_{p=0}^{\infty}\mathrm{ch}_p(F), \qquad \mathrm{ch}_p(F) := \mathrm{Tr}\frac{1}{p!}\left(\frac{\mathbf{i}F}{2\pi}\right)^p = \mathrm{Tr}e_p(F) \qquad (3)$$

with $\mathrm{ch}(F)$ the total Chern character. $e_p(F)$ and $\mathrm{ch}_p(F)$ are Hermitian, and $\mathrm{ch}_p(F)$ is a closed form locally written as $\mathrm{ch}_p(F) = d\omega_{2p-1}(A)$ with the Chern-Simons form $\omega_{2p-1}(A)$. We also use the following combination

$$\mathbf{K}(F) = e(F) - \frac{1}{n}\mathrm{ch}(F)\mathbf{1}_n, \qquad \mathbf{K}_p(F) = e_p(F) - \frac{1}{n}\mathrm{ch}_p(F)\mathbf{1}_n, \qquad (4)$$

which are traceless parts of $e(F)$ and $e_p(F)$, respectively. As we will see later, our self-duality relation defined below can be split into the relation between the pieces with respect to the decomposition by these basis.

In this paper, we call $\mathcal{E}_p^{(1)}$ and $\mathcal{E}_p^{(2)}$ single and double trace terms, respectively. We will not involve triple or higher trace terms. Although our model is not the most general in this sense, we will see that the double trace term is sufficient to be considered in order to support topological solitons.

We consider Bogomol'nyi equations derived from the pseudo-energy (1), which can be written in the form of a self-duality relation with respect to the Hodge dual operator $*$ induced by the metric as

$$\alpha_p\mathbf{K}_p(F) = *\epsilon_1\alpha_{n-p}\mathbf{K}_{n-p}(F), \qquad \tilde{\beta}_p\mathrm{ch}_p(F) = *\epsilon_2\tilde{\beta}_{n-p}\mathrm{ch}_{n-p}(F) \qquad (5)$$

where $p = 0, 1, \cdots, [n/2]$, $\epsilon_1^2 = \epsilon_2^2 = 1$ and we have defined

$$\tilde{\beta}_p^2 := \beta_p^2 + \alpha_p^2/n. \qquad (6)$$

Here we have used an equation

$$\mathrm{Tr}\mathbf{K}_p(F) \wedge *\mathbf{K}_p(F) = \mathrm{Tr}\left(e_p(F) - \frac{1}{n}\mathrm{ch}_p(F)\mathbf{1}_n\right) \wedge *\left(e_p(F) - \frac{1}{n}\mathrm{ch}_p(F)\mathbf{1}_n\right)$$

$$= \mathrm{Tr}e_p(F) \wedge *e_p(F) - \frac{1}{n}\mathrm{ch}_p(F) \wedge *\mathrm{ch}_p(F). \qquad (7)$$

In Eq.(5) $\epsilon_{1,2}$ denote signs which are deterimend in order that the Bogomol'nyi bound given below becomes positive. Eq.(5) is preserved by a simultaneous multiplication of a constant

$\lambda \, (\neq 0)$ on parameters in the pseudo-energy, $(\alpha_p, \alpha_{n-p}, \beta_p, \beta_{n-p}) \rightarrow (\alpha'_p, \alpha'_{n-p}, \beta'_p, \beta'_{n-p}) = \lambda(\alpha_p, \alpha_{n-p}, \beta_p, \beta_{n-p})$. In general, there are $[n/2]$ pairs of equations of this type and in such a generic case it is too hard to discuss about a nontrivial solution. Therefore we will restrict our attention to the case in which only one pair of equations exists. Let us restrict the integer p as $1 \leq p \leq [n/2]$ and suppose that nonzero parameters are only $\alpha_p, \alpha_q, \beta_p, \beta_q$ with $q := n - p$. Then our pseudo-energy has only four terms as

$$E_{p,q}[\alpha_p, \beta_p, \alpha_q, \beta_q; F] = \alpha_p^2 \mathcal{E}_p^{(1)} + \alpha_q^2 \mathcal{E}_q^{(1)} + \beta_p^2 \mathcal{E}_p^{(2)} + \beta_q^2 \mathcal{E}_q^{(2)}. \tag{8}$$

This pseudo-energy can be rewritten in terms of $\mathbf{K}_{p,q}(F)$ and $\mathrm{ch}_{p,q}(F)$ as

$$E_{p,q}[\alpha_p, \beta_p, \alpha_q, \beta_q; F] = \int_M \left\{ \alpha_p^2 \mathrm{Tr} \mathbf{K}_p(F) \wedge * \mathbf{K}_p(F) + \alpha_q^2 \mathrm{Tr} \mathbf{K}_q(F) \wedge * \mathbf{K}_q(F) \right.$$
$$\left. + \tilde{\beta}_p^2 \mathrm{ch}_p(F) \wedge * \mathrm{ch}_p(F) + \tilde{\beta}_q^2 \mathrm{ch}_q(F) \wedge * \mathrm{ch}_q(F) \right\}, \tag{9}$$

where coefficients $\tilde{\beta}_p$ and $\tilde{\beta}_q$ are defined in (6). This pseudo-energy exhibits the Bogomol'nyi completion

$$E_{p,q}[\alpha_p, \beta_p, \alpha_q, \beta_q; F] = \int_M \mathrm{Tr} \left(\alpha_p \mathbf{K}_p(F) - \epsilon_1 * \alpha_q \mathbf{K}_q(F) \right) \wedge * \left(\alpha_p \mathbf{K}_p(F) - *\epsilon_1 \alpha_q \mathbf{K}_q(F) \right)$$
$$+ \int_M \left(\tilde{\beta}_p \mathrm{ch}_p(F) - *\epsilon_2 \tilde{\beta}_q \mathrm{ch}_q(F) \right) \wedge * \left(\tilde{\beta}_p \mathrm{ch}_p(F) - *\epsilon_2 \tilde{\beta}_q \mathrm{ch}_q(F) \right)$$
$$+ 2\epsilon_1 \alpha_p \alpha_q \int_M \mathrm{Tr} \mathbf{K}_p(F) \wedge \mathbf{K}_q(F) + 2\epsilon_2 \tilde{\beta}_p \tilde{\beta}_q \int_M \mathrm{ch}_p(F) \wedge \mathrm{ch}_q(F). \tag{10}$$

Here the integrands in the first two lines are positive definite, and therefore the pseudo-energy is bounded from below as follows

$$E_{p,q}[\alpha_p, \beta_p, \alpha_q, \beta_q; F] \geq 2\epsilon_1 \alpha_p \alpha_q \int_M \mathrm{ch}_n(F) + 2 \left(\epsilon_2 \tilde{\beta}_p \tilde{\beta}_q - \frac{\epsilon_1 \alpha_p \alpha_q}{n} \right) \int_M \mathrm{ch}_p(F) \wedge \mathrm{ch}_q(F)$$
$$=: Q_p[\alpha_p, \beta_p, \alpha_q, \beta_q, \epsilon_1, \epsilon_2; F]. \tag{11}$$

The quantity $Q_p[\alpha_p, \beta_p, \alpha_q, \beta_q, \epsilon_1, \epsilon_2; F]$ is defined by the integration of an exact form over the whole space and is topological. The pseudo-energy is bounded from below by it, and the equality is saturated by configurations satisfying the Bogomol'nyi equations (5) as expected.

III.   A BPS SOLUTION ON THE PROJECTIVE SPACE

In this section we construct an explicit solution to the Bogomol'nyi equations (5) when the base space is $\mathbb{C}P^n$ with the Fubini-Study metric. Basics of this space used in this paper are summarized in Appendix. We work on a patch whose points are parameterized by an n-dimensional complex (column) vector W. The Fubini-Study metric and the corresponding Kähler two-form are given by

$$ds^2 = \frac{|dW|^2}{1+r^2} - \frac{|W^\dagger dW|^2}{(1+r^2)^2}, \qquad\qquad J := \frac{1}{4\mathbf{i}} d\left(\frac{W^\dagger dW - dW^\dagger W}{(1+r^2)}\right), \qquad (12)$$

respectively, with $r^2 = W^\dagger W$. We show that the following gauge field A_0 and the corresponding field strength F_0 give a solution to the Bogomol'nyi equations (5) if the parameters in the pseudo-energy satisfy certain relations:

$$A_0 = \frac{1}{r^2}(1 - \cos\rho)(W dW^\dagger - dW W^\dagger) + \frac{1}{2r^4}(1 - \cos\rho)^2 (W^\dagger dW - dW^\dagger W) W W^\dagger$$

$$F_0 := dA_0 + A_0 \wedge A_0 = V \wedge V^\dagger, \qquad (13)$$

where $\cos\rho = 1/\sqrt{1+r^2}$ and the one-form valued vector V denotes the corresponding vielbein, given as [see Eq. (A9)]

$$V = \cos\rho \, dW - \frac{1}{r^2}(1 - \cos\rho)\cos\rho \, W W^\dagger dW. \qquad (14)$$

The gauge field A_0 coincides with a part of the spin connection (A14) of the Fubini-Study metric, and the field strength F_0 coincides with the curvature 2-form R in Eq. (A16). In the previous section we consider the Hermitian matrix-valued differential form $\mathbf{K}_p(F)$ and the Chern character $\mathrm{ch}_p(F)$ for a field strength F. Let us substitute $F = F_0$ and consider quantities $\mathbf{K}_p(F_0)$ and $\mathrm{ch}_p(F_0)$. The field strength F_0 satisfies the following duality relations

$$*p\pi^p \mathbf{K}_p(F_0) = -(-1)^{n(n+1)/2} q\pi^q \mathbf{K}_q(F_0)$$

$$*\pi^p \mathrm{ch}_p(F_0) = (-1)^{n(n+1)/2}\pi^q \mathrm{ch}_q(F_0) \qquad (15)$$

as shown in (A27) in which the curvature 2-form R is replaced here by the field strength F_0. The field strength F_0 and the Kähler 2-form J in (12) satisfy [see Eq. (A26)]

$$\mathrm{Tr} F_0{}^{\wedge k} = -(2\mathbf{i})^k J^{\wedge k}. \qquad (16)$$

Comparing Eq. (15) with Eq. (5), the parameters in the pseudo-energy should be chosen as

$$\alpha_p = \lambda_1 p \pi^p, \qquad\qquad \alpha_q = \lambda_1 q \pi^q,$$

$$\tilde{\beta}_p = \lambda_2 \pi^p, \qquad\qquad \tilde{\beta}_q = \lambda_2 \pi^q. \tag{17}$$

When the parameters in the pseudo-energy satisfy these relations with certain parameters λ_1, λ_2, the connection A_0 can be a solution to the Bogomol'nyi equations with the signs

$$\epsilon_1 = -(-1)^{n(n+1)/2}, \qquad\qquad \epsilon_2 = (-1)^{n(n+1)/2}. \tag{18}$$

From Eqs. (6) and (17) the parameters β_p and β_q are obtained as

$$\beta_p = \sqrt{\tilde{\beta}_p^2 - \frac{\alpha_p^2}{n}} = \lambda_1 \sqrt{\mu^2 - \frac{p^2}{n^2}} \pi^p =: \lambda_3(p) \pi^p,$$

$$\beta_q = \sqrt{\tilde{\beta}_q^2 - \frac{\alpha_q^2}{n}} = \lambda_1 \sqrt{\mu^2 - \frac{q^2}{n^2}} \pi^q =: \lambda_3(q) \pi^q \tag{19}$$

with $\mu := \lambda_2/\lambda_1$. This implies that in the parameter region, $\mu^2 < q^2/n^2$, β_q becomes pure imaginary and the connection A_0 fails to be a solution to the Bogomol'nyi equations. The pseudo-energy (8) contains four free parameters $\alpha_p, \alpha_q, \beta_p, \beta_q$. The gauge field A_0 solves the Bogomol'nyi equations if these parameters satisfy the relations $\alpha_p/\alpha_q = p\pi^p/q\pi^q$ and $\beta_p/\beta_q = \pi^p/\pi^q$.

The topological charge of this configuration can be calculated, to give

$$Q_p[\lambda_1 p \pi^p, \lambda_3(p)\pi^p, \lambda_1 q \pi^q, \lambda_3(q)\pi^q, -(-1)^{n(n+1)/2}, (-1)^{n(n+1)/2}; F_0]$$

$$= -(-1)^{n(n+1)/2} 2\lambda_1^2 \pi^n \left\{ pq \int_M \mathrm{ch}_n(F) - \left(\mu^2 + \frac{pq}{n}\right) \int_M \mathrm{ch}_p(F) \wedge \mathrm{ch}_q(F) \right\}$$

$$= -(-1)^{n(n+1)/2} 2\lambda_1^2 \pi^n \left\{ pq \frac{1}{n!} \int_M \mathrm{Tr}\left(\frac{\mathbf{i}F_0}{2\pi}\right)^n - \left(\mu^2 + \frac{pq}{n}\right) \frac{1}{p!q!} \int_M \mathrm{Tr}\left(\frac{\mathbf{i}F_0}{2\pi}\right)^p \mathrm{Tr}\left(\frac{\mathbf{i}F_0}{2\pi}\right)^q \right\}$$

$$= (-1)^{n(n-1)/2} 2\lambda_1^2 \left\{ pq \frac{1}{n!} + \left(\mu^2 + \frac{pq}{n}\right) \frac{1}{p!q!} \right\} \int_M J^{\wedge n}$$

$$= 2\lambda_1^2 \left\{ pq \frac{1}{n!} + \left(\mu^2 + \frac{pq}{n}\right) \frac{1}{p!q!} \right\} n! \int_M dv$$

$$= 2\lambda_1^2 \left\{ pq + \left(\mu^2 + \frac{pq}{n}\right) {}_nC_p \right\} \frac{\pi^n}{n!}. \tag{20}$$

Here the relation (16) reduces the integrand in the third line into the power $J^{\wedge n}$ of the Kähler 2-form J. The resultant is proportional to the volume form dv of $\mathbb{C}P^n$.

Before closing this section we comment on the relation with self-duality relations on the six-dimensional sphere [10, 11]. In our previous work [10, 11], a connection, obtained by

embedding the spin connection ω of the standard metric on S^6 into 8×8 Hermitian matrices in terms of the Clifford algebra, solves similar self-duality equation; $\mathbf{K}_1[F_\omega] = \gamma_7 * \epsilon \mathbf{K}_2[F_\omega]$. Here F_ω is the corresponding field strength, the gauge group is SO(6), and γ_7 is the chirality matrix in the Clifford algebra. It gives a twisted closed form $\mathrm{ch}(F; \gamma_7) := \mathrm{Tr}\gamma_7 \boldsymbol{e}(F)$.

In a generalized gauge theory whose gauge group is unitary group U(n) we can consider such a twisted operator $\mathrm{ch}(F; \phi) := \mathrm{Tr}\phi \boldsymbol{e}(F)$ where ϕ is a covariantly constant operator $i.e.$ $D_A\phi := d\phi + [A, \phi] = 0$. The condition ensures that the differential form $\mathrm{ch}(F; \phi)$ is closed and its integration gives a topologically invariant quantity. For fixed x, $\phi(x)$ is a Hermitian matrix. We can diagonalize $\phi(x)$ with a unitary matrix $U(x)$; $\varphi(x) = U(x)\phi(x)U(x)^{-1}$. Therefore the equation becomes $d\varphi + [\tilde{A}, \varphi] = 0$ where $\tilde{A} = UAU^{-1} + UdU^{-1}$. In addition the commutator part $[\tilde{A}, \varphi]$ has no diagonal element. Therefore we conclude that φ is locally constant and it commutes with $\tilde{A}$; $[\tilde{A}, \varphi] = 0$. If the representation is irreducible as a representation of SU(n), such a matrix must be scalar operation and it becomes the type of Eq. (5). In the model on the six-dimensional sphere, which was commented above, γ_7 is constant and commute with all elements of SO(6) embedded into 8×8 Hermitian matrices. In this case, the matrix γ_7 is not a scalar matrix ; $\gamma_7 \neq c\mathbf{1}$ for any $c \in \mathbb{C}$. Actually this representation is reducible as a representation of the SO(6).

IV.   CONCLUSION AND DISCUSSION

In this paper, we have studied the generalized self-duality relations of U(n) gauge group and the pseudo-energies which lead them. After we discussed the equations on general $2n$ dimensional manifolds, we have constructed a model on the complex projective space $\mathbb{C}P^n$, the Bogomol'nyi equations on which are solved by the spin connection with respect to the Fubini-Study metric. The self-duality relations include double trace terms, implying that it does not solve the self-duality equations previously considered on S^6 which contain only single trace term [10, 11]. Originally these solutions were considered by Tchrakian $et.$ $al.$ [8]. We have revised it and have found non-vanishing pseudo-energy.

Our solution exists only when the parameters in the pseudo-energy (8) satisfy the relations $\alpha_p/\alpha_q = p\pi^p/q\pi^q$ and $\beta_p/\beta_q = \pi^p/\pi^q$. At this stage we do not know if another solution exists when these relations are not satisfied. However the Bogomol'nyi bound exists for any parameter region, and so there should exist a solution. Our solution satisfies the Bogomol'nyi

equations with definite signs for ϵ_1 and ϵ_2 given in Eq. (18). We have not yet known if a different sign asignment allows any solution. Our solution is topological in the sense that energy bound is saturated by a topological quantity, but we have not specified what kind of homotopy is related to this. We have constructed a solution with the minimum topological charge *i.e.* a single soliton. Multiple solitons and their dynamics deserve to be studied. Dimensional reductions to lower dimensions will introduce Higgs scalar fields. In this case there will be monopole-like solutions [9] too.

We could consider more general pseudo-energy of the form in Eq. (1) containing several terms in general more than four as considered in this paper. Moreover we have not considered multi-trace terms, so an extension including them remains as a future problem.

Finally it is interesting to apply the solution found in this paper to a compactification of higher dimension [8, 11] such as $AdS_4 \times \mathbb{C}P^3$ which may have a potential use in string theory [13].

APPENDIX A: DUALITY RELATIONS ON COMPLEX PROJECTIVE SPACES

In this appendix we summarize properties of the complex projective space such as higher order wedge products of the curvature 2-form and their Hodge dual, which will be used in this paper. The *n-dimensional complex projective space* $\mathbb{C}P^n$ is defined as a family of one-dimensional subspaces in an $(n+1)$-dimensional vector space $\mathbb{C}^{n+1}$. It can be written as a quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^\times$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. *Homogeneous coordinates* of $\mathbb{C}^{n+1}$ are used as representatives; $[Z_1 : \cdots : Z_{n+1}]$. An equivalence relation is introduced by scalar multiplications: $[\alpha Z_1 : \cdots : \alpha Z_{n+1}] = [Z_1 : \cdots : Z_{n+1}]$, $(\alpha \neq 0)$. The space $\mathbb{C}P^n$ is covered by $(n+1)$-patches U_i; $U_i := \{[Z_1 : \cdots : Z_{n+1}] \in \mathbb{C}P^n | Z_i \neq 0\}$. Each patch U_i is isomorphic to $\mathbb{C}^n$ where the isomorphism is given by $W_j^{(i)} = Z_j/Z_i$, $(i \neq j)$. On the patch U_{n+1} we can choose a local coordinate system $W = (W_1, \cdots, W_n)^t \in \mathbb{C}^n$ $(W_j := W_j^{(n+1)})$. Suppose that the space is endowed with the *Fubini-Study metric*, which is given as a line element on the

patch U_{n+1};

$$ds^2 = \frac{(1 + |W|^2)|dW|^2 - |W^\dagger dW|^2}{(1 + |W|^2)^2} \quad , \qquad\qquad |W|^2 = W^\dagger W \quad . \tag{A1}$$

Because of the symmetry, this can be rewritten in the same form in other patches. This space is not only a Riemannian but also an Hermitian manifold. It can be realized as a homogeneous space $\mathbb{C}P^n = \mathrm{U}(n + 1)/(\mathrm{U}(n) \times \mathrm{U}(1))$. The Fubini-Study metric can be identified with a part of the Maurer-Cartan form as seen later. We define the radial quantity $r = |W|$ and a function $\rho(r) := \arctan r$. The corresponding Lie algebra $u(n + 1)$ is a set consisting of anti-Hermitian matrices. The algebra $u(n+1)$ split into two parts; $u(n) \oplus u(1)$ and $cp(n)$.

$$u(n) \oplus u(1) := \{\mathbf{i}X \in u(n + 1)|[\phi, X] = 0\} \ ,$$

$$cp(n) := \{\mathbf{i}X \in u(n + 1)|\{\phi, X\} = 0\} \ ,$$

$$\phi := \begin{pmatrix} \mathbf{1}_n & \\ & -1 \end{pmatrix} \quad , \tag{A2}$$

where $\mathbf{1}_n$ is the $n \times n$ unit matrix and the matrix X is traceless and Hermitian. Note that each element in the subspace $u(n) \oplus u(1)$ is block-diagonal $A \oplus B$ where A is an $n \times n$ matrix and B is a complex number. The matrix $\varphi = \mathrm{diag}(\mathbf{1}_n, -n)$ [1] is a generator of the $u(1)$ part. A generator in $cp(n)$ is represented by a matrix with the following form:

$$X = \begin{pmatrix} & W \\ W^\dagger & \end{pmatrix} \quad . \tag{A3}$$

We often use the normalized matrix $\hat{X} = \frac{1}{|W|}X$ for convenience. The exponential of this generator becomes an element in group $\mathrm{U}(n + 1)$.

$$g = \exp(\mathbf{i}\rho\hat{X}) = \sum_{k=0}^{\infty} \frac{1}{k!}(\mathbf{i}\rho\hat{X})^k = \mathbf{1}_{n+1} + (\cos\rho - 1)\hat{X}^2 + \mathbf{i}\sin\rho\hat{X} \quad . \tag{A4}$$

where we have inserted a scale factor ρ which we leave arbitrary in order to obtain the Fubini-Study metric. Here we have used

$$\hat{X}^2 = \begin{pmatrix} \hat{W}\hat{W}^\dagger & \\ & 1 \end{pmatrix} \quad , \qquad \hat{X}^3 = \hat{X} \ . \tag{A5}$$

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[1] The matrix φ differs from ϕ.

The inverse of g is its Hermitian conjugate $g^\dagger$: $g^\dagger g = 1$. The *Maurer-Cartan form* α_{MC} is defined as

$$\alpha_{\mathrm{MC}} = g^{-1}dg = (1 - \cos\rho)(\hat{X}d\hat{X} - d\hat{X}\hat{X}) + (\cos\rho - 1)^2\hat{X}^2 d\hat{X}\hat{X}$$
$$+ \mathbf{i}\left(\hat{X}d\rho + \sin\rho d\hat{X} + (1 - \cos\rho)\sin\rho\hat{X}d\hat{X}\hat{X}\right) \quad . \tag{A6}$$

This takes a value in the Lie algebra $u(n+1)$ splitting into two parts with respect to the decomposition $u(n+1) = (u(n) \oplus u(1)) \oplus cp(n)$: $\alpha_{\mathrm{MC}} = \tilde{\omega} + \mathbf{i}E$ with

$$\tilde{\omega} = (1 - \cos\rho)(\hat{X}d\hat{X} - d\hat{X}\hat{X}) + (\cos\rho - 1)^2\hat{X}^2 d\hat{X}\hat{X} \in u(n) \oplus u(1),$$
$$E = \hat{X}d\rho + \sin\rho d\hat{X} + (1 - \cos\rho)\sin\rho\hat{X}d\hat{X}\hat{X} \in cp(n) \quad . \tag{A7}$$

The matrix valued one-form E gives the vielbein V with respect to the Fubini-Study metric whereas the $u(n) \oplus u(1)$ part gives the corresponding spin connection. First the vielbein is obtained from E by

$$E = \begin{pmatrix} 0_n & V \\ V^\dagger & 0_n \end{pmatrix} \tag{A8}$$

to yield

$$V = \cos\rho dW - \frac{1}{r^2}(1 - \cos\rho)\cos\rho WW^\dagger dW \quad , \qquad \cos\rho = \frac{1}{\sqrt{1 + r^2}} \quad . \tag{A9}$$

The Fubini-Study metric can be obtained by the inner product of $V^\dagger$ and V with the symmetric product as

$$ds^2 = V^\dagger \cdot V = \sum_i V_i^* V_i = \frac{|dW|^2}{1 + r^2} - \frac{|W^\dagger dW|^2}{(1 + r^2)^2} \quad , \tag{A10}$$

and the wedge product of them gives the Kähler two-form

$$J := \frac{1}{2\mathbf{i}} V^\dagger \wedge V = \frac{1}{2\mathbf{i}}\left(\frac{dW^\dagger \wedge dW}{1 + r^2} - \frac{dW^\dagger WW^\dagger \wedge dW}{(1 + r^2)^2}\right) \quad . \tag{A11}$$

This form is closed and at least on the patch U_{n+1} it can be written as an exact form:

$$J = \frac{1}{4\mathbf{i}} d\left(\frac{W^\dagger dW - dW^\dagger W}{(1 + r^2)}\right) \quad , \tag{A12}$$

where this equation does not mean that the form J is exact all around the space $\mathbb{C}P^n$. The form J is a nontrivial closed form, so the equivalence class including the form J is a

non-zero element of the cohomology ring $H^*(\mathbb{C}P^n) = \mathrm{Ker}(d : \Lambda^*(\mathbb{C}P^n) \to \Lambda^*(\mathbb{C}P^n))/\mathrm{Im}(d : \Lambda^*(\mathbb{C}P^n) \to \Lambda^*(\mathbb{C}P^n))$. Second, we consider the $u(n) \oplus u(1)$ part $\tilde{\omega}$, which is a spin connection with respect to the Fubini-Study metric. As mentioned above, generators in the $u(n) \oplus u(1)$ part are block-diagonal. We separate them into an $n \times n$ matrix-valued one-form ω and a one-form ξ with pure imaginary coefficient,

$$\tilde{\omega} = \begin{pmatrix} \omega & 0 \\ 0 & \xi \end{pmatrix} , \tag{A13}$$

$$\omega = \frac{1}{r^2}(1 - \cos\rho)(W\,dW^\dagger - dW\,W^\dagger) + \frac{1}{2r^4}(1 - \cos\rho)^2(W^\dagger dW - dW^\dagger W)WW^\dagger ,$$

$$\xi = \frac{1}{2}\cos^2\rho\,(W^\dagger dW - dW^\dagger W) . \tag{A14}$$

Here the form ξ is a gauge potential of the Kähler two-form J as shown in (A12). They form actually a spin connection:

$$dV + \omega \wedge V + V \wedge \xi = 0 . \tag{A15}$$

The curvature 2-form R obtained from the $n \times n$ block ω is computed as

$$R := d\omega + \omega \wedge \omega = V \wedge V^\dagger , \qquad \mathrm{Tr}R = -V^\dagger \wedge V = -(2\mathbf{i})J . \tag{A16}$$

Next we calculate powers of the curvature R and their Hodge dual. The above relation (A16) shows that the exponents of the curvature R are written as

$$R^{\wedge s} = (2\mathbf{i})^{s-1}J^{\wedge s-1} \wedge R . \tag{A17}$$

In order to calculate its Hodge dual, we define the Hodge dual on forms. Let us decompose the vielbein V in (A9) in the real space;

$$V_i = \sigma_i + \mathbf{i}\tau_i , \qquad ds^2 = \sum_i \{(\sigma_i)^2 + (\tau_i)^2\} . \tag{A18}$$

Here σ_i and τ_i are differential forms with real coefficients, that is to say, $\sigma_i^\dagger = \sigma_i$, $\tau_i^\dagger = \tau_i$. They are vielbeins in the real space. The invariant volume form with respect to the Fubini-Study metric can be written by the real vielbeins as

$$dv := \sigma_{1\cdots n} \wedge \tau_{1\cdots n} = \sigma_1 \wedge \sigma_2 \cdots \wedge \sigma_n \wedge \tau_1 \wedge \tau_2 \cdots \wedge \tau_n , \tag{A19}$$

integration of which gives the volume. The metric is expressed as a matrix G and its determinant is denoted by $g = \det G$. Then we have

$$\beta = \frac{1}{1 + r^2} \ , \qquad\qquad\qquad G = \beta \mathbf{1}_n - \beta^2 W W^\dagger \ ,$$

$$GW = \beta W - \beta^2 r^2 W = \beta^2 W \ , \qquad\qquad GU = \beta U \ , (W^\dagger U = 0). \qquad (A20)$$

The eigenvalues of G are $n-1$ βs and β^2. The determinant g is $g = \beta^{2(n+1)}$, and the volume is

$$\mathrm{vol}(\mathbb{C}P^n) = \int dv = \frac{\pi^n}{n!}. \qquad (A21)$$

Let us introduce a new notation to express σ and τ in the same form: $\mathfrak{e}_i := \sigma_i$, $\mathfrak{e}_{n+i} := \tau_i$, $1 \le I_k \le 2n$, $1 \le k \le s$, $\mathfrak{e}_{I_1 I_2 \cdots I_s} := \mathfrak{e}_{I_1} \wedge \mathfrak{e}_{I_2} \wedge \cdots \wedge \mathfrak{e}_{I_s}$. The Hodge dual operator $*$ on a basis of s-forms is defined as

$$*\mathfrak{e}_{I_1 \cdots I_s} = \frac{1}{(2n-s)!} \epsilon_{I_1 \cdots I_s I_{s+1} \cdots I_{2n}} \mathfrak{e}_{I_{s+1} \cdots I_{2n}} \ . \qquad (A22)$$

We are ready to compute the Hodge duals of J, R and their powers. We rewrite J and R by σ and τ. The Kähler two-form J and the coefficients of the kth-powers of the curvature R are written as

$$V_i^\dagger \wedge V_i = 2\mathbf{i}\sigma_i \wedge \tau_i \ , \qquad J = \sum_{i=1}^n \sigma_i \wedge \tau_i \ ,$$

$$\left(R^{\wedge k}\right)_{ij} = (2\mathbf{i})^{k-1} J^{\wedge k-1} \wedge (\sigma_i \wedge \sigma_j + \tau_i \wedge \tau_j - \mathbf{i}\sigma_i \wedge \tau_j + \mathbf{i}\tau_i \wedge \sigma_j) \ . \qquad (A23)$$

The Hodge duals of powers of the Kähler two-form J and components of powers of the curvature are summarized:

$$*J^{\wedge k} = (-1)^{n(n-1)/2} \frac{k!}{(n-k)!} J^{\wedge n-k} \ ,$$

$$*\left(J^{\wedge k} \wedge \sigma_{ij}\right) = -(-1)^{n(n-1)/2} \frac{k!}{(n-k-2)!} J^{\wedge n-k-2} \wedge \tau_{ij},$$

$$*\left(J^{\wedge k} \wedge \tau_{ij}\right) = -(-1)^{n(n-1)/2} \frac{k!}{(n-k-2)!} J^{\wedge n-k-2} \wedge \sigma_{ij},$$

$$*\left(J^{\wedge k} \wedge \sigma_i \wedge \tau_j\right) = -(-1)^{n(n-1)/2} \left\{ \frac{k!}{(n-k-2)!} \times J^{\wedge n-k-2} \wedge (\sigma_j \wedge \tau_i) \right.$$

$$\left. - \delta_{ij} \frac{k!}{(n-k-1)!} J^{\wedge n-k-1} \right\}. \qquad (A24)$$

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The Hodge duals of the coefficients of $R^{\wedge k}$ satisfy the following relations:

$$* \left(R^{\wedge k} \right)_{ij} = -(-1)^{n(n-1)/2} \frac{(k-1)!}{(n-k-1)!} \frac{(2\mathbf{i})^{k-1}}{(2\mathbf{i})^{n-k-1}} \left(R^{\wedge n-k} \right)_{ij}$$
$$- (-1)^{n(n-1)/2} \delta_{ij} (2\mathbf{i})^k \frac{(k-1)!}{(n-k)!} J^{\wedge n-k}$$
$$*\mathrm{Tr}R^{\wedge k} = -(-1)^{n(n-1)/2} \frac{(k-1)!}{(n-k-1)!} \frac{(2\mathbf{i})^{k-1}}{(2\mathbf{i})^{n-k-1}} \mathrm{Tr}R^{\wedge n-k}$$
$$- (-1)^{n(n-1)/2} n (2\mathbf{i})^k \frac{(k-1)!}{(n-k)!} J^{\wedge n-k} \tag{A25}$$

Since the k-th exponent of J is proportional to $\mathrm{Tr}R^{\wedge k}$ we have

$$\mathrm{Tr}R^{\wedge k} = -(2\mathbf{i})^k J^{\wedge k} \quad . \tag{A26}$$

Finally, by using notations defined in Eqs.(3) and (4), the relations in Eq.(A25) are summarized as

$$\epsilon := (-1)^{n(n+1)/2}$$
$$*k\pi^k \mathbf{K}_k(R) = -\epsilon(n-k)\pi^{n-k} \mathbf{K}_{n-k}(R)$$
$$*\pi^k \mathrm{ch}_k(R) = \epsilon\pi^{n-k} \mathrm{ch}_{n-k}(R). \tag{A27}$$

where field strength F in Eqs.(3) and (4) are replaced by the curvature 2-form R.

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